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On vector bundles and chiral matter in N=1 heterotic compactifications

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Abstract: In this note we derive the net number of generations of chiral fermions in heterotic string compactifications on Calabi-Yau threefolds with $SU(n)$ vector bundles, for n odd, using the parabolic approach for bundles. This extends the work of Friedman, Morgan and Witten, who treated $SU(n)$ vector bundles, with n even, which lead to a vanishing net amount of chiral matter. We then compare our results with the spectral cover construction for bundles and make a comment on the net number interpretation in F-theory.

Keywords: Superstrings and Heterotic Strings, F-Theory, Differential and Algebraic Geometry.

JHEP01(1999)011

Contents

JHEP01(1999)011

1. Introduction

Recent progress has been made in our understanding of heterotic string compactification on Calabi-Yau threefold Z with vector bundle V embedded in $E_8 \times E_8$ [1] as well as in its expected dual F-theory compactification on Calabi-Yau fourfold X [2, 3, 4] which both lead to $N = 1$ supersymmetry in four dimensions. Part of the progress was the appearance (independent of any duality consideration) of non-perturbative heterotic five-branes n_5 which are necessary for a consistent heterotic compactification [1]. On the F-theory side, the necessity of turning on a number n_3 of space-time filling three-branes for tadpole cancellation [5] which match the heterotic five-branes [1, 6] was established. Further, it was shown that this matching can be refined due to the occurrence of some discrete data on the heterotic -and F-theory side [7]. In particular, it is expected that the associated moduli spaces of the two theories are isomorphic [1, 4, 7, 8, 9, 10, 11].

Apart form the matching of moduli in both theories, which lead to massless gauge neutral chiral matter, one should also understand the part of the 4D heterotic spectrum which corresponds to massless under the gauge group charged matter as well as chiral matter in terms of geometry on the F-theory side. Progress in this direction was made in [4, 12, 13, 14, 15, 16, 17, 18].

Besides the spectrum matching, further checks of het/F-theory duality involve the comparision of $N = 1$ effective interactions which are determined by holomorphic quantities, i.e. superpotentials and gauge kinetic functions [19, 20, 21, 22].

Let us be more concrete! The heterotic/F-theory duality picture was established by considering Z as an elliptic fibration $\pi : Z \to B_2$ with a section σ, where B_2 is a two-fold base; Z can be represented as a smooth Weierstrass model. X was considered as being elliptically fibered over a threefold base B_3, which is rationally ruled, i.e. a fibration $B_3 \to B_2$ with P^1 fibers exists because we have to assume that the fourfold is a $K3$ fibration over the twofold base B_2 in order to extend adiabatically the 8D duality between the heterotic string on T^2 and F-theory on $K3$ over the base B_2.

In addition to the specification of Z, we have to specify a stable vector bundle over Z, which breaks part of the $E_8 \times E_8$ gauge symmetry. We will consider a $G = SU(n)$ vector bundle which determines a rank n complex vector bundle V of trivial determinant. Actually, there are three methods for constructing stable vector bundles over elliptic fibrations: the parabolic, the spectral cover and the construction via del Pezzo surfaces, which are explained and developed in [1] and [23, 24, 25]. We will adopt the parabolic construction here, since it allows us to easily compute Chern-classes, however we will compare our results to those obtained from the spectral cover construction. In particular it will be shown that both approaches agree for a certain choice of the twisting line bundle on the spectral cover.

In the parabolic approach [1] considered a component of the moduli space of $SU(n)$ bundles (understood as rank n complex vector bundles) which are τ-invariant, i.e. the involution of the elliptic fibration $Z \to B$ lifts up to V. This condition could be implemented for $SU(n)$ bundles with n even.

In the spectral cover approach [1] considered $V = SU(n)$ bundles with n arbitrary. These bundles possess an additional degree of freedom since one has the possibility to 'twist' with a line bundle $\mathcal{N}$ on the spectral cover C, which leads to a multi-component structure of the moduli space of such bundles. Further, they did not require any τ-invariance of V in the construction, moreover, they found τ-invariance for a certain class of these bundles which have no additional twists. In particular, they found the same modulo conditions (see below) as in the parabolic approach, in order to get τ-invariance. Further, their τ-invariant bundles have vanishing third Chern-class.

In section 2.1 we will first review, for our purposes, some facts of the parabolic construction for V an $SU(n)$ vector bundle, then we will recall what is known for n-even which was discussed in [1] and we will extend this to the case if n is odd. In contrast to [1] we do not focus on a τ-invariant point in the moduli space of $SU(n)$ bundles, this allows us to determine the net number N_{gen} of generations of chiral fermions, i.e. ($\sharp$ generations $-$ $\sharp$ antigenerations) in the observable sector of the 4D unbroken gauge group. In contrast to the bundles at the τ-invariant point, which have n even and a modulo 2 condition for η, our bundles have n odd and $\pi_*(c_2(V)) = \eta$ divisible by n. Then we will proceed in section 2.2 and compare them with the spectral cover construction of V and in section 2.3 conclude with a comment on certain discrete data in F-theory which turn out to be related to N_{gen}.

We will close with an enlarged discussion in section 2.4 on what we have shown!

JHEP01(1999)011

2. Bundle construction

2.1 Parabolic approach

In the parabolic approach we start with an unstable bundle on a single elliptic curve E which is given by[1]

$$V = W_k \oplus W_{n-k}^* \, ; \tag{2.1}$$

this has the property that it can be deformed by an element $\alpha \in H^1(E, W_k^* \otimes W_{n-k}^*)$ to a (semi) stable bundle V' over E which fits in the exact sequence

$$0 \to W_{n-k}^* \to V' \to W_k \to 0 \, . \tag{2.2}$$

Now, to get a global version of this construction, we are interested in an unstable bundle over Z, which reduces on every fibre of $\pi : Z \to B$ to the unstable bundle over E and which can be deformed to a stable bundle over Z. Since the basic bulding blocks were on each fibre W_k with $W_1 = \mathcal{O}(p)$ we have to replace them by their global versions. So, one replaces global $W_1 = \mathcal{O}(p)$ by $W_1 = \mathcal{O}(\sigma)$ and W_k can be defined inductively by an exact sequence

$$0 \to \mathcal{L}^{n-1} \to W_k \to W_{k-1} \to 0 \, , \tag{2.3}$$

with $\mathcal{L} = K_B^{-1}$. Further, one has globally the possibility to twist by additional data coming from the base B and so one can write [1] for $V = SU(n)$

$$V = W_k \otimes \mathcal{M} \oplus W_{n-k}^* \otimes \mathcal{M}' \, . \tag{2.4}$$

Note that the unstable bundle can be deformed to a stable one by an element in $H^0(B, R^1\pi_*(ad(V)))$ since the Leray spectral sequence degenerates to an exact sequence [1]. Further, note that we can use the unstable bundle for the computation of characteristic classes, because the topology of the bundle is invariant under deformations.

Now, let us start with an unstable $G = SU(n)$ bundle V where $\mathcal{M}, \mathcal{M}'$ are line bundles over B which are constrained so that V has trivial determinant, i.e. $\mathcal{M}^k \otimes (\mathcal{M}')^{n-k} \otimes \mathcal{L}^{-\frac{1}{2}(n-1)(n-2k)} \cong \mathcal{O}$. Further, W_k and W_{n-k}^* are defined as

$$W_k = \bigoplus_{a=0}^{k-1} \mathcal{L}^a \, , \qquad W_{n-k}^* = \bigoplus_{b=0}^{n-k-1} \mathcal{L}^{-b} \, , \tag{2.5}$$

where we have set $\mathcal{L}^0 = \mathcal{O}(\sigma)$ and $\mathcal{L}^{-0} = \mathcal{O}(\sigma)^{-1}$ with $\mathcal{L}$ being a line bundle on B. The total Chern class of V can be written as

$$c(V) = \prod_{a=0}^{k-1} \left(1 + c_1(\mathcal{L}^a) + c_1(\mathcal{M})\right) \prod_{b=0}^{n-k-1} \Big(1 + c_1(\mathcal{L}^{-b}) + c_1(\mathcal{M}')\Big) \, . \tag{2.6}$$

[1] here W_k can be defined inductively as a unique non-split extension $0 \to \mathcal{O} \to W_{k+1} \to W_k \to 0$ (c.f. [1])

Now we will discuss the following two cases: n is even and n is odd. First let us review the case that n is even which was considered in [1], then we will extend to the case n is odd and compare both cases with the spectral cover construction of V.

n **even** In this case, one can choose $k = \frac{n}{2}$ which restricts one to take $\mathcal{M}' = \mathcal{M}^{-1}$ in order to obey the trivial determinant of V. The advantage of taking $k = \frac{n}{2}$ is that the condition of τ invariance of V is easily implemented. Note that τ operates on V as $\tau^* V = V^*$, i.e. $k \to n-k$. Now, the expansion of the total Chern class of V immediatly leads to $c_1(V) = 0$ and $c_3(V) = 0$. Further setting $\sigma = c_1(\mathcal{O}(\sigma))$ and $\eta = -2c_1(\mathcal{M}) + c_1(\mathcal{L})$ respectively using the fact that $\sigma^2 = -\sigma c_1(\mathcal{L})$, one obtains for the second Chern class

$$c_2(V) = \eta\sigma - \frac{1}{24} c_1(\mathcal{L})^2 (n^3 - n) - \frac{n}{8} \eta(\eta - nc_1(\mathcal{L})) \,. \tag{2.7}$$

Moreover, from $c_1(\mathcal{M}) = -\frac{1}{2}(\eta - c_1(\mathcal{L}))$ one gets the congruence relation for η:

$$\eta \equiv c_1(\mathcal{L}) \qquad (mod\, 2) \tag{2.8}$$

Thus $c_2(V)$ is uniquely determined in terms of η and the elliptic Calabi-Yau manifold Z. In particular one has

$$\eta = \pi_*(c_2(V)) \,. \tag{2.9}$$

Now, let us turn to the case that n is odd!

n **odd** Let us first specify our unstable bundle V. Actually we have n different choices to do this which depend on the choice of the integer k in the range $1 \leq k \leq n$. We will choose $k = \frac{n+1}{2}$ and the line bundles $\mathcal{M} = \mathcal{S}^{\frac{-n+1}{2}}$ and $\mathcal{M}' = \mathcal{S}^{\frac{n+1}{2}} \otimes \mathcal{L}^{-1}$ which will presently be shown to be appropriate in order to be compared with results obtained from spectral covers for V. Therefore we can write for our unstable bundle V

$$V = W_k \otimes \mathcal{S}^{-n+k} \oplus W^*_{n-k} \otimes \mathcal{S}^k \otimes \mathcal{L}^{-1} \,, \tag{2.10}$$

which has trivial determinant. Using the above relation for the total Chern-class of V and setting again $\sigma = c_1(\mathcal{O}(\sigma))$ and $\sigma^2 = -\sigma c_1(\mathcal{L})$, and

$$\eta = nc_1(\mathcal{S}) \tag{2.11}$$

we will find for the characteristic classes of V

$$c_1(V) = 0 \,, \tag{2.12}$$
$$c_2(V) = \eta\sigma - \frac{1}{24} c_1(\mathcal{L})^2 (n^3 - n) - \frac{n}{8} \eta(\eta - nc_1(\mathcal{L})) + \frac{1}{8n} \eta \, (\eta - nc_1(\mathcal{L})) \,, \tag{2.13}$$
$$c_3(V) = \frac{1}{n} \sigma\eta(\eta - nc_1(\mathcal{L})) \,, \tag{2.14}$$

So, we are restricted to bundles V with $\eta = \pi_*(c_2(V))$ divisible by n.

Now, the integration of c_3 over Z can be accomplished by first integrating over the fibers of $Z \to B$ and then integrating over the base. Further, using the fact that the section σ intersects the fiber F in one point $\sigma \cdot F = 1$ and $r = 3\sigma$ where r is the cohomology class dual to the vanishing of the section of the line bundle $\mathcal{O}(1)$ (defined on the total space of the Weierstrass model of Z), we get

$$\int_Z c_3(V) = \int_B \frac{1}{n}\eta(\eta - nc_1(\mathcal{L}))\,. \tag{2.15}$$

Further recall, that the net number of generations N_{gen} of chiral fermions in the observable sector of the gauge group is given by (c.f. [26])

$$N_{gen} = \frac{1}{2}\left|\int_Z c_3(V)\right|, \tag{2.16}$$

which reflects the fact that massless fermions in four dimensions correspond to zero modes of the Dirac operator on the Calabi-Yau manifold Z. So we get

$$N_{gen} = n_{gen} - \bar{n}_{gen} = \int_B \frac{1}{2n}\eta(\eta - nc_1(\mathcal{L})) \tag{2.17}$$

and thus, if we fix the elliptic manifold Z, so that the section σ and $c_1(\mathcal{L})$ are fixed, N_{gen} is uniquely determined for a choice of η. Here we have to note that in order to separately determine the number of generations n_{gen} respectively, antigenerations $\bar{n}_{gen}$ instead of just their difference, we have to compute in addition the dimension of $H^1(Z,V)$. This can be done by using the Leray spectral sequence.

Remarks Using N_{gen} we can write for the total number of $N = 1$ chiral matter multiplets which are charged under the unbroken gauge group $C^c_{het} = N_{gen} + 2\bar{n}_{gen}$. Recall that the number of $N = 1$ neutral chiral (resp. antichiral) multiplets C_{het} is given by (c.f.[10]) $C_{het} = h^{21}(Z) + h^{11}(Z) + m_{bun}$ with m_{bun} denoting the number of bundle moduli, i.e. the dimension of $H^1(Z, ad(V))$. Note, in the former case of $V = SU(n)$ with n even, which was discusse in [1], we have $C^c_{het} = 2\bar{n}_{gen}$.

2.2 Comparison with the spectral cover construction

In order to compare our results with those obtained from the spectral cover construction let us review some facts of the setup [1].

The spectral cover C is given by the vanishing of a section s of $\mathcal{O}(\sigma)^n \otimes \mathcal{M}$ with $\mathcal{M}$ being an arbitrary line bundle over B of $c_1(\mathcal{M}) = \eta'$. The locus $s = 0$ for the section is given for n even by $s = a_0 z^n + a_2 z^{n-2}x + a_3 z^{n-3}y + ... + a_n x^{n/2}$ (resp. the last term is $x^{(n-3)/2}y$ for n odd)[2]. Further, one has a twisting line bundle $\mathcal{N}$ over

[2]here $a_r \in \Gamma(B, \mathcal{M}\otimes\mathcal{L}^{-r})$, a_0 is a section of $\mathcal{M}$ and x, y sections of $\mathcal{L}^2$ resp. $\mathcal{L}^3$ in the Weierstrass model (c.f. [1])

JHEP01(1999)011

C so that we have the vector bundle $V = \pi_{2*}(\mathcal{N} \otimes \mathcal{P}_B)$ over Z with π_2 being the projection of $C \times_B Z$ to the second factor. Now, since the Poincare line bundle $\mathcal{P}_\mathcal{B}$ becomes trivial when restricted to σ and with Grothendieck-Riemann-Roch for the projection $C \to B$, we get

$$\pi_*(e^{c_1(\mathcal{N})} Td(C)) = ch(V) Td(B)\,, \tag{2.18}$$

and with the condition $c_1(V) = 0$ one has

$$c_1(\mathcal{N}) = -\frac{1}{2}(c_1(C) - \pi_* c_1(B)) + \gamma = \frac{1}{2}(n\sigma + \eta' + c_1(\mathcal{L})) + \gamma\,; \tag{2.19}$$

here $\gamma \in H^{1,1}(C, \mathbf{Z})$ with $\pi_*\gamma = 0 \in H^{1,1}(B, \mathbf{Z})$. In particular if one denotes by K_B and K_C the canonical bundles of B and C then one has (c.f. [1])

$$\mathcal{N} = K_C^{1/2} \otimes K_B^{-1/2} \otimes \left(\mathcal{O}(\sigma)^n \otimes \mathcal{M}^{-1} \otimes \mathcal{L}^n\right)^\lambda, \tag{2.20}$$

from which one learns that $\gamma = \lambda(n\sigma - \eta' + nc_1(\mathcal{L}))$.

The second Chern class of V is given by

$$c_2(V) = \eta'\sigma - \frac{1}{24} c_1(\mathcal{L})^2 \left(n^3 - n\right) - \frac{n}{8}\eta' \left(\eta' - nc_1(\mathcal{L})\right) - \frac{1}{2}\pi_* \left(\gamma^2\right) \tag{2.21}$$

where the last term reflects the fact that one can twist with a line bundle $\mathcal{N}$ on the spectral cover, one has

$$\pi_*(\gamma^2) = -\lambda^2 n\eta' \left(\eta' - nc_1(\mathcal{L})\right). \tag{2.22}$$

Now let us compare!

In case that n is even it was shown [1] that to achieve τ invariance in the spectral cover approach for V, one must define $\mathcal{N}$ in the above sense with $\gamma = 0$, i.e. $\lambda = 0$. In particular it was shown the existence of an isomorphism $\mathcal{N}^2 = K_C \otimes K_B^{-1}$. Further it was shown that there are the same mod two coditions for η' and n in the spectral cover approach for $\gamma = 0$ and in particular that also $\eta' = \pi_*(c_2(V))$ and therefore one is lead to the identification $\eta' = \eta$.

Also in case that n is even and $\lambda = \frac{1}{2}$ the last two terms in (2.21) combine (c.f. [1], [7]) the only general elements of $H^{1,1}(C, \mathbf{Z})$ are $\sigma|_C$ and $\pi^*\beta$ (for $\beta \in H^{1,1}(B, \mathbf{Z})$), which have because of $C = n\sigma + \pi^*\pi_* c_2 V$ the relation $\pi_*(\sigma|C) = \pi_*\sigma(n\sigma + \pi^*\eta) = \pi_*\sigma(-nc_1 + \pi^*\eta) = \eta - nc_1$; so $\gamma = \lambda(n\sigma - \pi^*(\eta - nc_1))$ (with λ possibly half-integral) and $\pi_*(\gamma^2) = -\lambda^2 n\eta(\eta - nc_1)$; so for $\lambda = 1/2$ the term would disappear.

Now if n is odd, we can identify the last term in (2.13) with $\frac{\pi_*(\gamma^2)}{2}$ if we choose $\lambda = \frac{1}{2n}$, i.e. the parabolic approach for n odd agrees with the spectral cover approach if we choose the twisting line bundle $\mathcal{N}$ appropriate on the spectral cover. Furthermore, if we use

$$c_1(\mathcal{N}) = \frac{1}{2}\left(n\sigma + \eta' + c_1(\mathcal{L})\right) + \gamma = \frac{(n+1)}{2}\sigma + c_1(\mathcal{L}) + \frac{(n-1)}{2}\frac{\eta'}{n}, \tag{2.23}$$

which is well defined for n odd and since we can choose $\mathcal{M} = \mathcal{S}^n$ we are left with $\eta' = nc_1(\mathcal{S})$ and so with $\lambda = \frac{1}{2n}$ we have the same conditions for η' and n as we had in the parabolic approach.

2.3 A comment on some discrete data

Let us now recall, it was shown [28] that one has as quantization law for the four-flux $G = \frac{1}{2\pi} dC$ the modified integrality condition $G = \frac{c_2}{2} + \alpha$ with $\alpha \in H^4(X, \mathbf{Z})$. Furthermore, it was shown [7] that α is further restricted by $\int \alpha^2 + \alpha c_2 \leq -120$ in order to keep the wanted amount of supersymmetry in a consistent compactification. It has also been shown [27] that the appearance of the four-flux modifies the number of space-time filling threebranes n_3 which are necessary for tadpole cancellation in F-theory [5]; one has

$$n_3 = \frac{\chi(X^4)}{24} - \frac{1}{2}G^2 . \tag{2.24}$$

Now a recent paper showed [7] that on the heterotic side the additional degree of freedom, coming from possible twists of the line bundle $\mathcal{N}$ on the spectral cover modifies the number of heterotic fivebranes

$$n_5(\gamma) = n_5(\gamma = 0) + \frac{1}{2}\pi_* \left(\gamma^2\right) . \tag{2.25}$$

Moreover, [7] showed that the heterotic "twists" correspond to the appearance of a non-trivial four-flux on the F-theory side

$$\pi_*(\gamma^2) = -G^2 , \tag{2.26}$$

and therefore rounded off the picture that the number of heterotic fivebranes matches the number of F-theory threebranes which was established in [1] for E_8 and in [6] for $SU(n)$ vector bundle.

Now, using the identification of $\frac{1}{8n}\eta(\eta - nc_1(\mathcal{L}))$ with $\frac{1}{2}\pi_*(\gamma^2)$ at $\lambda = \frac{1}{2n}$, denoting this by $\frac{1}{2}\pi_*(\gamma^2)|_{\lambda=1/2n} = \frac{1}{2}\pi_*(\gamma_n^2)$, we can write the third Chern class at $\lambda = \frac{1}{2n}$

$$\int_B c_3(V) = 4 \int_B \pi_* \left(\gamma_n^2\right) . \tag{2.27}$$

and therefore see that N_{gen} is related to the appearance of the four-flux in F-theory.[3]

2.4 Discussion

We have constructed a class of $SU(n_{odd})$ vector bundles using the parabolic bundle construction. This extends the work of Friedman, Morgan and Witten [1], who treated a class of $SU(n_{even})$ vector bundles in the parabolic approach.

[3] At this point I am very grateful to G. Curio for many inspiring discussions and inparticular for pointing out such a relation to me.

For n even and $\eta \equiv c_1(\mathcal{L})(\bmod\ 2)$, the bundles in the parabolic construction of [1] are restricted to a subset, given at $\lambda = 0$, of τ-invariant bundles in the spectral cover construction. We have shown that for n odd and $\eta \equiv 0(\bmod\ n)$, the bundles in the parabolic construction are restricted to a subset, given at $\lambda = \frac{1}{2n}$, of bundles in the spectral cover construction having no τ-invariance.

In contrast to the $SU(n_{even})$ bundles, which have $c_3(V) = 0$ and so do not lead to a net amount of chiral matter, the $SU(n_{odd})$ bundles have $c_3(V) \neq 0$ and lead to a net amount of chiral matter.

Furthermore, we have shown that the net number of generations is directly related to the 4-flux in F-theory for $SU(n_{odd})$ bundles in the parabolic construction or equivalently to bundles fixed at the locus $\lambda = \frac{1}{2n}$ in the spectral cover construction (see note added for a extended explanation on that point). Thus the net amount of chiral matter gets a direct interpretation on the F-theory side in purely geometrical terms.

Note added In a recent paper [29], the computation of $c_3(V)$ was performed in the spectral cover approach, it is given by (in the $l = 0$ sector c.f. [29])

$$\int_B c_3(V) = 2\int_B \lambda\eta(\eta - nc_1)\,. \tag{2.28}$$

For $\lambda = \frac{1}{2n}$, this is in nice agreement with our computation of $c_3(V)$ in the parabolic approach (2.15) and therefore supports our results. However, it shows also that away from the $\lambda = \frac{1}{2n}$ locus the relation (2.27) gets lost and therefore the relation between 4-flux and net amount of chiral matter too. Thus the bundles at $\lambda = \frac{1}{2n}$ are excellent for heterotic string /F-theory considerations.

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